

Non-minimal couplings, quantum geometry and black hole entropy

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Abstract

The black hole entropy calculation for type I isolated horizons, based on loop quantum gravity, is extended to include non-minimally coupled scalar fields. Although the non-minimal coupling significantly modifies quantum geometry, the highly non-trivial consistency checks for the emergence of a coherent description of the quantum horizon continue to be met. The resulting expression of black hole entropy now depends also on the scalar field precisely in the fashion predicted by the first law in the classical theory (with the same value of the Barbero-Immirzi parameter as in the case of minimal coupling).

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I. INTRODUCTION

In classical general relativity, weakly isolated horizons provide a unified framework to analyze properties of black hole and cosmological horizons in equilibrium [1, 2, 3, 4], where the geometry and matter fields on the horizon itself are assumed to be time independent but the physics in the exterior region can be dynamical. Although the horizons typically lie in a highly curved region of space-time, their symmetry groups fall in to three universality classes [3]. Cases of direct physical interest are the type I horizons where the *intrinsic* geometry and matter fields *on* the horizon are spherical and type II horizons where they are axi-symmetric. Note that these symmetries need not extend in the exterior region; a class of Robinson-Trautman solutions provide an explicit example where the isolated horizon is of type I but where the 4-geometry does not admit a Killing field in any neighborhood of the horizon [5].

The sector of general relativity consisting of space-times with a weakly isolated horizon inner boundary admits an action principle and a Hamiltonian formulation. It is therefore possible to carry out canonical quantization of this sector. For type I horizons this procedure was implemented in detail in [6, 7, 8]. The implementation required an extension of the bulk quantum geometry [9-22] to accommodate the presence of a boundary, and the construction of a $U(1)$ Chern-Simons theory to describe the geometry of the quantum horizon. The requirement that the inner boundary is an isolated horizon is incorporated in the quantum theory by promoting the horizon boundary condition to an operator equation. This allows the horizon to fluctuate but requires that the fluctuations be correlated in a way dictated by the classical boundary condition. The form of this *quantum horizon condition* is such that a coherent theory can emerge if and only if eigenvalues of a certain operator in the quantum theory of the bulk geometry are *exactly* equal to those of another operator in the surface Chern-Simons theory. This is a stringent requirement because the two theories are quite independent and eigenvalues of each operator can be computed in the respective theory without any knowledge of the other! Yet, the boundary conditions introduced in the isolated horizon framework relate the parameters appearing in the two theories in just the right way for the equality to hold (see section II).

Next, one can construct a micro-canonical ensemble by fixing the horizon area and charges and calculate the number of microstates in the ensemble. However, the bulk quantum geometry has a 1-parameter ambiguity, labelled by what is known as the ‘Barbero-Immirzi parameter’ $\gamma > 0$. This has close similarity with the θ -ambiguity in QCD [22]. There are no physical operators mixing states in distinct γ -sectors; there is super-selection. As with the parameter θ in QCD, the value of γ in Nature is to be determined by experiments. States and operators in various γ sectors are very similar in their structure but the eigenvalues of geometrical operators scale with γ . Hence the γ ambiguity trickles down to the expression of the number of horizon states. Irrespective of the value of γ , the entropy turns out to be proportional to the horizon area a_o . However, the coefficient depends on the value of γ . Since there is a single undetermined parameter, its value can be fixed by a single ‘experiment’—for example, by measuring the smallest eigenvalue of the area operator. Unfortunately, technology necessary for a direct measurement of this type is not available. But we can use the Bekenstein-Hawking semi-classical entropy formula

$$S = \frac{1}{4 \ell_{\text{Pl}}^2} a_o, \quad (1.1)$$

where, ℓ_{Pl} is the Planck length, to ‘carry out an indirect measurement’. Suppose, for

example, that we demand that the leading term in the expression of entropy of a large Schwarzschild black hole be given by (1.1). This fixes the value of γ ,

$$\gamma = \frac{\ln 2}{\pi\sqrt{3}} \quad (1.2)$$

and hence the theory. One can now test this theory. In particular, is the entropy of charged black holes or of cosmological horizons correctly recovered in this theory? In [8], the answer was shown to be in the affirmative for type I —i.e., non-rotating, undistorted— horizons. That work has since been extended to type II horizons, i.e., to incorporate rotation and distortion [23].

These calculations incorporated the possible presence of Maxwell and scalar fields at the horizon (possibly with dilatonic couplings), allowing for non-zero electric, magnetic and dilatonic charges. However all these fields are *minimally coupled to gravity*. Now, using Killing horizons, Jacobson, Kang and Myers [24] and Iyer and Wald [25] showed that in presence of *non-minimal* couplings to gravity, the first law of black hole mechanics in classical general relativity is non-trivially modified, suggesting that the entropy should now depend not only on the area but also on the values of matter fields at the horizon. For non-minimally coupled scalar fields, this analysis was recently extended to the isolated horizon framework [26]. Specifically, in the theory governed by the action:¹

$$\mathcal{S}[g_{ab}, \phi] = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G} f(\phi) R - \frac{1}{2} g^{ab} \partial_a \phi \partial_b \phi - U(\phi) \right], \quad (1.3)$$

where R is the scalar curvature of the metric g_{ab} and U is a potential for the scalar field, the entropy is given by [26]

$$S = \frac{1}{4\ell_{Pl}^2} \oint_S f(\phi) d^2V \quad (1.4)$$

where S is any 2-sphere cross-section of the horizon. A natural question now is: Can the quantum geometry calculation incorporate this situation? At first sight, this seems to be difficult because matter fields at the horizon play no essential role in that calculation; the calculation is dictated almost entirely by the geometry of the quantum horizon.

Now, in the case when f is nowhere vanishing, one could first re-express the classical theory using the ‘Einstein frame’ by an appropriate conformal rescaling of the metric that removes the non-minimal coupling and *then* carry out quantization as in [7, 23]. However, that procedure would simply ‘by-pass’ the issue, rather than meeting it ‘head-on’, leaving the ramifications of non-minimal coupling unexplored. In the classical theory, one can carry out the entire analysis in either the non-minimally coupled Jordan frame or minimally coupled Einstein frame and *demonstrate* that results agree [26]. Can one do the same in the quantum theory? Already at the classical level, a priori, it is not obvious that the agreement must hold (see section V of [26]). At the quantum level, it is even less clear that the stringent requirements for the emergence of a coherent description of the quantum horizon can be satisfied in the Jordan frame. Finally, the derivation of the first law in [26] was carried out

¹ Here we have ignored the surface term. In loop quantum gravity, one uses a first order framework based on tetrads and connections. The first order action for this theory, including the surface term, is given in [26].

using a first order action which is equivalent to (1.3) if f is nowhere vanishing. However, in the first order formalism f can be allowed to vanish on open sets which remain bounded away from the horizon and infinity; the first law still holds, with entropy given by (1.4). In the first order framework, in general it is not even possible to pass to the Einstein frame and a direct analysis in the Jordan frame is necessary.

In this paper, for simplicity, we will restrict ourselves to type I isolated horizons and confront the non-minimal coupling directly in the Jordan frame using the first order framework of [26]. We will find that the presence of non-minimal coupling introduces a major modification in the quantum theory of the bulk geometry and also changes the ‘level’ (i.e. the coupling constant) of the surface Chern Simons theory. But the two modifications conspire to leave the delicate matching between the bulk and horizon quantum structures in tact, whence a coherent theory of the geometry of the quantum horizon continues to exist also in the Jordan frame. One can then calculate entropy. One now finds that for large black holes the entropy is given by (1.4) (rather than (1.1)) for the same value of the Barbero-Immirzi parameter γ as in the case of the minimal coupling.

II. MINIMALLY COUPLED MATTER: SUMMARY

To stream-line the calculation and to bring out the modifications brought about by non-minimal coupling, we will first recall the highlights of the analysis in the case when all fields are minimally coupled to gravity. For brevity, we will overlook subtleties, some of which are conceptually important. These are discussed in detail in [8].

In loop quantum gravity, one begins with a Hamiltonian formulation of general relativity. The configuration variable A_a^i is an $SU(2)$ connection on a 3-manifold M and the momentum variable is represented by a 2-form field Σ_{ab}^i which takes values in the Lie algebra of $SU(2)$. A_a^i is a spin-connection on M and $E_i^a := \gamma \eta^{abc} \Sigma_{bc}^i$ has the physical interpretation of an orthonormal triad of density weight 1, where $\gamma > 0$ is the Barbero-Immirzi parameter and η^{abc} is the metric independent, density weighted Levi-Civita 3-form on M .

Let us focus on the sector of the theory consisting of space-times which admit a type I isolated horizon Δ with a fixed area a_o as the internal boundary. Then M is asymptotically flat and has an internal boundary S , topologically a 2-sphere, the intersection of M with Δ . Introduce on S an internal, unit, radial vector field r^i (i.e. any isomorphism from the unit 2-sphere in the Lie algebra of $SU(2)$ to S). Then it turns out that *the intrinsic geometry of S is completely determined by the pull-back $\underline{A}^i r_i =: W$ to S of the (internal-radial component of the) connection A^i on M* [4]. Furthermore, W is in fact a spin-connection intrinsic to the 2-sphere S . Finally, the fact that S is (the intersection of M with) a type I isolated horizon is captured in a relation between the two canonically conjugate fields:

$$F : \hat{\equiv} dW \hat{\equiv} - \frac{2\pi\gamma}{a_o} \underline{\Sigma}^i r_i. \quad (2.1)$$

where $\underline{\Sigma}^i$ is the pull-back to S of the 2-forms Σ^i on M . (Throughout, $\hat{\equiv}$ will stand for equality restricted to Δ .) Thus, because of the isolated horizon boundary conditions, fields which would otherwise be independent are now related. In particular, the pull-backs to S of our canonically conjugate fields A^i , Σ^i are completely determined by the $U(1)$ connection W .

In absence of an internal boundary, the symplectic structure is given just by a volume integral [27]. In presence of the internal boundary under consideration, it now acquires a

surface term [7, 8]:

$$\Omega(\delta_1, \delta_2) = \frac{1}{8\pi G} \left[\int_M \text{Tr} (\delta_1 A \wedge \delta_2 \Sigma - \delta_2 A \wedge \delta_1 \Sigma) + \frac{a_o}{\gamma\pi} \oint_S \delta_1 W \wedge \delta_2 W \right], \quad (2.2)$$

where $\delta \equiv (\delta A, \delta \Sigma)$ denotes tangent vectors to the phase space Γ . Since W is essentially the only ‘free data’ on the horizon, it is not surprising that the surface term of the symplectic structure is expressible entirely in terms of W . However, it is interesting that the new surface term is precisely the symplectic structure of the U(1)-Chern Simons theory. *The symplectic structures of the Maxwell, Yang-Mills, scalar and dilatonic fields do not acquire surface terms and, because of minimal coupling, do not feature in the gravitational symplectic structure either.* Conceptually, this is an important point: this, in essence, is the reason why the black hole entropy depends just on the horizon area and not, in addition, on the matter charges [8].

One can systematically ‘quantize’ this sector of the phase space [8]. We can focus only on the gravitational field since the matter fields do not play a significant role. One begins with a Kinematic Hilbert space $\mathcal{H} = \mathcal{H}_V \otimes \mathcal{H}_S$ where \mathcal{H}_V is the Hilbert space of states in the bulk [10, 11, 12, 13, 14] and \mathcal{H}_S is the Hilbert space of surface states. Expression (2.2) of the symplectic structure implies that \mathcal{H}_S should be the Hilbert space of states of the Chern-Simons theory on the punctured S , where the ‘level’, or the coupling constant, is given by:

$$k = \frac{a_o}{4\pi\gamma\ell_{\text{Pl}}^2} \quad (2.3)$$

A pre-quantization consistency requirement is that k be an integer [8].

Our next task is to encode in the quantum theory the fact that Δ is a type I horizon with area a_o . This is done by imposing the horizon boundary condition (2.1) as an *operator equation*:

$$(1 \otimes \hat{F}) \Psi \hat{\equiv} - \left(\frac{2\pi\gamma}{a_o} (\hat{\Sigma} \cdot r) \otimes 1 \right) \Psi, \quad (2.4)$$

on admissible states Ψ in \mathcal{H} . Now, a general solution to (2.4) can be expanded out in a basis: $\Psi = \sum_n \Psi_V^{(n)} \otimes \Psi_S^{(n)}$, where $\Psi_V^{(n)}$ is an eigenvector of the ‘triad operator’ $-(2\pi\gamma/a_o)(\hat{\Sigma} \cdot r)(x)$ on \mathcal{H}_V and $\Psi_S^{(n)}$ is an eigenvector of the ‘curvature operator’ $\hat{F}(x)$ on \mathcal{H}_S with same eigenvalues. Thus, the theory is non-trivial only if a sufficiently large number of eigenvalues of the two operators coincide. Since the two operators act on entirely different Hilbert spaces and are introduced quite independently of one another, this is a *very* non-trivial requirement.

Now, in the bulk Hilbert space \mathcal{H}_V , the eigenvalues of the ‘triad operator’ are given by [19]:

$$- \left(\frac{2\pi\gamma}{a_o} \right) \left(8\pi\ell_{\text{Pl}}^2 \sum_I m_I \delta^3(x, p_I) \eta_{ab} \right), \quad (2.5)$$

where m_I are half integers and η_{ab} is the natural, metric independent Levi-Civita density on S and p_I are points on S at which the polymer excitations of the bulk geometry in the state Ψ_V puncture S . A completely independent calculation [8], involving just the surface Hilbert space \mathcal{H}_S , yields the following eigenvalues of $\hat{F}(x)$:

$$\frac{2\pi}{k} \sum_I n_I \delta^3(x, p_I) \equiv 2\pi \frac{4\pi\gamma\ell_{\text{Pl}}^2}{a_o} \sum_I n_I \delta^3(x, p_I) \quad (2.6)$$

where n_I are integers modulo k . Thus, with the identification $-2m_I = n_I \bmod k$, the two sets of eigenvalues match exactly. Note that in the Chern-Simons theory the eigenvalues of $F(x)$ are dictated by the ‘level’ k and the isolated horizon boundary conditions tie it to the area parameter a_o just in the way required to obtain a coherent description of the geometry of the quantum horizon.

In the classical theory, the parameter a_o in the expression of the surface term of the symplectic structure (2.2) and in the boundary condition (2.1) is the horizon area. However in the *quantum theory*, a_o has simply been a parameter so far; we have not tied it to the *physical area of the horizon*. Therefore, in the entropy calculation, to capture the intended physical situation, one constructs a suitable ‘micro-canonical’ ensemble. This leads to the last essential technical step.

Let us begin by recalling that, in quantum geometry, the area eigenvalues are given by [18, 19]

$$8\pi\gamma\ell_{\text{Pl}}^2 \sum_I \sqrt{j_I(j_I + 1)}.$$

We can therefore construct a micro-canonical ensemble by considering only that sub-space of the volume theory which, at the horizon, satisfies:

$$a_o - \epsilon \leq 8\pi\gamma\ell_{\text{Pl}}^2 \sum_I \sqrt{j_I(j_I + 1)} \leq a_o + \epsilon \quad (2.7)$$

where I ranges over the number of punctures, j_I is the spin label associated with the puncture p_I [8, 19].² In presence of matter fields carrying charges, we fix values of horizon charges Q_o and restrict the bulk matter states so that

$$Q_o - \epsilon' \leq Q_{\text{hor}} \leq Q_o + \epsilon' \quad (2.8)$$

for suitably chosen ϵ 's (one for each charge). Finally, to obtain entropy, we have to calculate the number of *surface states* in this ensemble, i.e., in the sub-space of \mathcal{H} in which the quantum boundary conditions and Einstein's equations are satisfied and in which the bulk states satisfy the condition spelled out in (2.7). The number is given by:

$$\mathcal{N} = \sum_p \sum_{j_1, \dots, j_p} \prod_{I=1}^p (2j_I + 1) \quad (2.9)$$

where the number p of punctures and the spin-labels j_1, \dots, j_p are chosen to satisfy the area constraint above. Through detailed analysis [8], one can estimate the right side of (2.9) and calculate the entropy of large black holes:

$$S_\Delta := \ln \mathcal{N} = \frac{\gamma_o}{\gamma} \frac{a_o}{4\ell_{\text{Pl}}^2} + o\left(\frac{\ell_{\text{Pl}}^2}{a_o}\right), \quad \text{where} \quad \gamma_o = \frac{\ln 2}{\pi\sqrt{3}} \quad (2.10)$$

Here, $o(\ell_{\text{Pl}}^2/a_o)$ denote terms which, when multiplied by ℓ_{Pl}^2/a_o approach zero in the limit a_o tends to infinity. Thus, the leading order contribution to the entropy is indeed proportional

² The appearance of the parameter ϵ is standard in statistical mechanics. It has to be much smaller than the macroscopic parameters of the system but larger than level spacings in the spectrum of the operator under consideration. Its precise value is irrelevant and does not affect the leading contribution to entropy. We require $8\sqrt{3}\pi\gamma\ell_{\text{Pl}}^2 < \epsilon \ll a_o$.

to the horizon area. However, even for large black holes, one obtains agreement with the Hawking-Bekenstein formula *only* in the sector of quantum geometry in which the Barbero-Immirzi parameter γ takes the value $\gamma = \gamma_o$. Thus, while all γ sectors are equivalent classically, the standard quantum field theory in curved space-times is recovered in the semi-classical theory only in the γ_o sector of quantum geometry. It is noteworthy that thermodynamic considerations involving *large* black holes can be used to fix the quantization ambiguity which dictates such Planck scale properties as eigenvalues of geometric operators. As noted in section I, the value of γ can be fixed by demanding agreement with the semi-classical result just in one case —e.g., a spherical horizon with zero charge, or a cosmological horizon in the de Sitter space-time, or, Once the value of γ is fixed, the theory is completely determined and in that theory, agreement with the Bekenstein-Hawking result holds for *all* isolated horizons with minimally coupled matter fields.

III. NON-MINIMAL COUPLING

Let us now turn to the non-minimally coupled scalar field discussed in [26]. Since we are considering type I horizons, the scalar field ϕ is time-independent and spherically symmetric — hence, constant— on Δ . Here, we wish to focus on the sector of the theory in which the inner boundary is a type I isolated horizon with area a_o and scalar field ϕ_o .

The detailed analysis of [26] is based on a first order action. One can carry out a Legendre transformation of that action and pass to the real canonical variables analogous to those used in section II by a γ -dependent canonical transformation. However, because of the non-minimal coupling, the symplectic structure is now modified. In place of (2.2) we have³

$$\begin{aligned} \Omega(\delta_1, \delta_2) = & \frac{1}{8\pi G} \int_M \text{Tr} [\delta_1 A \wedge \delta_2 (f(\phi)\Sigma) - \delta_2 A \wedge \delta_1 (f(\phi)\Sigma)] \\ & + \int_M K(\phi) [\delta_1 \phi \delta_2 (\star d\phi) - \delta_2 \phi \delta_1 (\star d\phi)] + \frac{a_o f(\phi_o)}{\gamma \pi} \oint_S [\delta_1 W \wedge \delta_2 W], \end{aligned} \quad (3.1)$$

where $f(\phi)$ is the function responsible for the non-minimal coupling in the action (1.3) and $K(\phi)$ is an algebraic function of ϕ , given by:

$$K(y) = [1 + (3/16\pi G)(f'(y))^2/f(y)]. \quad (3.2)$$

The classical analysis requires that $f(\phi)$ be non-zero in a neighborhood of S and of infinity and for definiteness we will assume that it is positive there. The form of terms in (3.1) has two interesting implications. First, the form of the gravitational bulk term tells us that the momentum Π_{ab}^i conjugate to the gravitational connection is given by

$$\Pi_{ab}^i = f(\phi)\Sigma_{ab}^i, \quad (3.3)$$

whence the orthonormal triad E_i^a of density weight 1 is now given by

$$E^{ai} = \gamma [f(\phi)]^{-1} \eta^{abc} \Pi_{bc}^i \quad (3.4)$$

³ At first it may appear that there is a discrepancy of a factor of -2 in the first term of this symplectic structure and the one in [26]. Note, however, that in [26] trace is performed in the Lie algebra of the Lorentz group while here the group is SU(2) and the -2 arises from the relation between the two.

Thus, *the Riemannian geometry of M is no longer dictated just by the momentum canonically conjugate to the gravitational connection but depends also on the scalar field.* This is a striking, qualitative difference from the case when one has only minimally coupled matter fields. Next, let us consider the surface term. Since it does not contain variations of ϕ , there is no surface symplectic structure for the scalar field whence, as in section II, the surface Hilbert space will continue to describe quantum states only of the horizon geometry. However, a major difference is that the coefficient of the surface term now involves the value ϕ_o of the scalar field on the horizon. Consequently, *the quantum horizon geometry will now depend on ϕ_o .*

The total kinematic Hilbert space \mathcal{H} again has the form $\mathcal{H} = \mathcal{H}_V \otimes \mathcal{H}_S$. States in the volume Hilbert space \mathcal{H}_V now describe not only the polymer excitations of the geometry but also the excitations of the scalar field (which reside at vertices of graphs at which the geometry is excited) [28, 29]. Because of the form of the surface term in the symplectic structure, the surface Hilbert is again the space of states of the U(1) Chern-Simons theory on the punctured S (with an arbitrary number of punctures). The level, which is dictated by the coefficient of the surface term in (3.1), is given by

$$k = \frac{a_o f(\phi_o)}{4\pi\gamma\ell_{\text{Pl}}^2}; \quad (3.5)$$

it now depends on the horizon value of the scalar field on S .

Next, let us consider the horizon boundary condition. In terms of geometric fields, it is again (2.1). However, to promote it to the quantum theory, we need to first express it in terms of the momentum conjugate to A_a^i and therefore now depends also on the scalar field:

$$F : \widehat{\equiv} dW \widehat{\equiv} - \frac{2\pi\gamma}{a_o f(\phi_o)} \underline{\Pi}^i r_i. \quad (3.6)$$

To encode in the quantum theory the fact that Δ is a type I isolated horizon with area a_o and scalar field ϕ_o , we now demand that states Ψ must satisfy

$$(1 \otimes \hat{F}) \Psi = - \left(\frac{2\pi\gamma}{a_o f(\phi_o)} (\hat{\underline{\Pi}} \cdot r) \otimes 1 \right) \Psi. \quad (3.7)$$

As before, a ‘sufficient number’ of solutions exist if and only if the bulk and the surface operators in this condition have a large number of common eigenvalues. Since Π_{ab}^i is the momentum conjugate to A_a^i , its eigenvalues can be read off from bulk quantum geometry and are the same as before, whence in place of (2.5), the eigenvalues of the bulk operator are now given by:

$$- \left(\frac{2\pi\gamma}{a_o f(\phi_o)} \right) \left(8\pi\ell_{\text{Pl}}^2 \sum_I m_I \delta^3(x, p_I) \eta_{ab} \right), \quad (3.8)$$

The eigenvalues of \hat{F} are dictated by the ‘level’ k of the Chern-Simons theory, which is now given by (3.5). Therefore, the eigenvalues of the surface operator are given by:

$$\frac{2\pi}{k} \sum_I n_I \delta^3(x, p_I) \equiv 2\pi \frac{4\pi\gamma\ell_{\text{Pl}}^2}{a_o f(\phi_o)} \sum_I n_I \delta^3(x, p_I) \quad (3.9)$$

Thus, again the eigenvalues agree when n_I are integers modulo k : Although the scalar field enters the quantum geometry operators in the bulk and the level of the surface Chern Simons theory because of non-minimal coupling, the two effects compensate each other precisely and the delicate balance between the volume and the surface theories required for the emergence of a coherent description of the quantum horizon persists.

Using this description, we can calculate entropy as before. The main differences are: i) we have to incorporate the scalar field in the construction of the micro-canonical ensemble, and ii) now the area operator is built from the gravitational momentum *and* the scalar field.

Starting from the expression of the area function a_S on the classical phase space and noting that $f(\phi)$ is a constant, $f(\phi_o)$, when restricted to S , we can repeat the procedure of [19] to introduce an area operator \hat{A}_S on \mathcal{H}_V . Its eigenvalues are now given by:

$$\frac{8\pi\gamma\ell_{\text{Pl}}^2}{f(\phi_o)} \sum_I \sqrt{j_I(j_I + 1)}. \quad (3.10)$$

where j_I are half integers and, as before, I label punctures made by the polymer excitations at the horizon. Let us denote by $\mathcal{H}_{V,a_o,\epsilon}^{\text{geometry}}$ the subspace of the geometry states in the bulk spanned by eigenvectors of the area operator \hat{A}_S with eigenvalues

$$a_o - \epsilon \leq \frac{8\pi\gamma\ell_{\text{Pl}}^2}{f(\phi_o)} \sum_I \sqrt{j_I(j_I + 1)} \leq a_o + \epsilon. \quad (3.11)$$

Next, let us consider the bulk states of the scalar field. In the polymer framework [28, 29], the quantum scalar field resides at vertices of graphs and can take continuous values at each vertex. Since the scalar field takes the value ϕ_o on S *on the entire phase space*, we will restrict ourselves to those bulk states which are eigenvectors of the scalar field operators $\hat{\phi}(p_I)$ associated with the punctures p_I on S where the eigenvalues lie in a small interval⁴ around ϕ_o :

$$\phi_o - \epsilon' \leq \phi_I \equiv \phi(p_I) \leq \phi_o + \epsilon'. \quad (3.12)$$

Denote this sub-space by $\mathcal{H}_{V,\phi_o,\epsilon'}^{\text{scalar}}$. Since there is no surface term in the symplectic structure for the scalar field, there is no surface Hilbert space for the scalar field. However, since the ‘level’ of the Chern-Simons theory depends on the value ϕ_o of the scalar field at the horizon, the surface Hilbert space of geometry now depends on ϕ_o .

Using these notions, we can now construct the micro-canonical ensemble. It consists of states in $\mathcal{H} = \mathcal{H}_V \otimes \mathcal{H}_S$ which: i) satisfy the quantum horizon boundary conditions; ii) for which the ‘volume part’ of the states lies in the subspace $\mathcal{H}_{V,a_o,\epsilon}^{\text{geometry}} \otimes \mathcal{H}_{V,\phi_o,\epsilon'}^{\text{scalar}}$; and, iii) which satisfy the quantum Einstein’s equations. Thus, the overall procedure is the same as in the minimally coupled case. For reasons explained in detail in [8], the entropy is given by the logarithm of the number \mathcal{N} of *surface states* in this ensemble in the sense described below.

Quantum Einstein’s equations can be imposed following the same procedure as in the minimally coupled case. As before, these are a set of three constraints. The implementation of the Gauss and the diffeomorphism constraints is the same as in [8]. The first says that

⁴ ϵ' is distinct from ϵ because whereas the spectrum of the area operator is discrete, that of the scalar field operator is continuous. Physical considerations suggest that ϵ' be constrained through: $0 < \epsilon'$ and $|f(\phi_o \pm \epsilon') - f(\phi_o)| < 8\sqrt{3\pi\gamma\ell_{\text{Pl}}^2}/a_o$.

the ‘total’ state in \mathcal{H} be invariant under the $SU(2)$ gauge rotations of triads and, as in [8], this condition is automatically met when the state satisfies the quantum boundary condition (3.7). The second constraint says that two states are physically the same if they are related by a diffeomorphism. The detailed implementation of this condition is rather subtle because an extra structure is needed in the construction of the surface Hilbert space and the effect of diffeomorphisms on this structure has to be handled carefully [8]. However, as in the minimally coupled case, the final result is rather simple: For surface states, what matters is only the number of punctures; their location is irrelevant. The last quantum constraint is the Hamiltonian one. As in the non-minimally coupled case, in the classical theory, the constraint is differentiable on the phase space only if the lapse goes to zero on the boundary. Therefore, this constraint restricts only the volume states. However, as in the minimally coupled case, there is an indirect restriction on surface states which arises as follows. Consider a set (p_I, j_I) with $I = 1, 2, \dots, N$ consisting of N punctures p_I and half-integers j_I real satisfying (3.11). We will refer to this set as ‘surface data’. Suppose there exists a bulk state satisfying the Hamiltonian constraint which is compatible with this ‘surface data’ and *some* choice of real numbers ϕ_I satisfying (3.12). Then, we can find compatible surface states such that the resulting states in the total Hilbert space \mathcal{H} lie in our ensemble. The space $S_{(p_I, j_I)}$ of these surface states is determined entirely by the surface data. In our state counting, we include the number $\mathcal{N}_{(p_I, j_I)}$ of these surface states.⁵ If, on the other hand, there is no bulk state satisfying the Hamiltonian constraint which is compatible with this ‘surface data’, then the surface states in $S_{(p_I, j_I)}$ will not appear in our ensemble and will be excluded in the counting. The total number \mathcal{N} of states responsible for the black hole entropy is obtained by adding up $\mathcal{N}_{(p_I, j_I)}$ corresponding to each $S_{(p_I, j_I)}$ admitted in our ensemble.

As in the minimally coupled case, we now have to make a mild assumption: we will assume that *given generic surface data, there is at least one bulk state which satisfies the Hamiltonian constraint for some choice of ϕ_I satisfying (3.12)*. Under this assumption —on which we comment below— the counting can be done as in [8], and one finds:

$$S_\Delta := \ln \mathcal{N} = \frac{\gamma_o}{\gamma} \frac{f(\phi_o) a_o}{4 \ell_{\text{Pl}}^2} + o\left(\frac{\ell_{\text{Pl}}^2}{a_o}\right), \quad \text{where } \gamma_o = \frac{\ln 2}{\pi \sqrt{3}} \quad (3.13)$$

where, as before, $o(\ell_{\text{Pl}}^2/a_o)$ denote terms which, when multiplied by ℓ_{Pl}^2/a_o approach zero as a_o tends to infinity. Thus, the entropy now depends on the scalar field and for isolated horizons with large a_o one recovers the classically expected expression (1.4) using *the same value* γ_o of the Barbero Immirzi parameter as in the minimally coupled case.

We will conclude with a few remarks.

1. Even though there is no complete quantum gravity theory, the calculation of black hole entropy has been possible in certain approaches because one can encase all the difficult issues pertaining to full quantum dynamics in a plausible assumption. In string theory, one assumes that non-perturbative effects such as the interactions between branes and anti-branes can be neglected; in the symmetry based approaches a la Carlip one assumes that

⁵ Note that there may be a large number —possibly infinite— of bulk states which are compatible with a given ‘surface data’ in this sense. This number does not matter because the bulk states are ‘traced out’ in calculating the entropy of the horizon. What matters for the entropy calculation is only the dimensionality of $S_{(p_I, j_I)}$.

certain transformations will be admissible symmetries of the full quantum theory; and, in quantum geometry one assumes that, for generic ‘surface data’, the bulk Hamiltonian constraint will admit at least one solution. However, there is a difference between the minimally coupled fields discussed in [8] and non-minimal couplings considered here. In the minimally coupled case the assumption was that for each choice of the set $\{j_I\}$ such that the *total* area lies in the range $(a_o - \epsilon, a_o + \epsilon)$, there is a bulk state satisfying the Hamiltonian constraint. Now, we assume that a solution to the (coupled) Hamiltonian constraint exists for some choice of the boundary values ϕ_I of ϕ , where *each* ϕ_I is in the range $(\phi_o - \epsilon', \phi_o + \epsilon')$. Is this too stringent a requirement? Should one instead assume only that an ‘average’ of the ϕ_I (perhaps weighted suitably by the j_I) equal ϕ_o ? It is clear that our requirement is a better quantum representation of the fact that, classically, $\phi|_S = \phi_o$ on the *entire* phase space. However, would the Hamiltonian constraint admit solutions where *at each puncture* ϕ_I is close to ϕ_o ? There are two reasons suggesting that the answer is in the affirmative. First, since we have the same number of constraints but more fields now, it should be easier to satisfy the quantum constraints in the bulk. This is certainly the case for classical constraints. Second, since the spectrum of $\hat{\phi}$ is continuous, by allowing ϕ_I to lie in the interval $(\phi_o - \epsilon', \phi_o + \epsilon')$ we are letting the bulk state to lie in an *infinite* dimensional sub-space of $\mathcal{H}_{V,\phi}$ at each puncture. Therefore, our assumption on the existence of solutions to the Hamiltonian constraints seems to be rather weak. Indeed, a priori, because of the first point, it seems weaker than that in the minimally coupled case. To summarize, it seems plausible to assume that isolated horizons with large areas, satisfying the field equations and the condition $\phi|_S = \phi_o$ of our classical phase space will be modelled by states in our micro-canonical ensemble.

2. There are two ‘polymer representations’ for the scalar field. In the first $\hat{\phi}(x)$ has well defined action, admits any real number as an eigenvalue and all its eigenvectors are normalizable. In the second representation, $\hat{\pi}(x)$, the field canonically conjugate to $\hat{\phi}$, has these nice properties but $\hat{\phi}(x)$ are not well-defined; only $\widehat{\exp(i\mu\phi(x))}$ are well-defined for arbitrary real numbers μ [29]. The first representation is better suited in cases where the scalar field is non-minimally coupled and/or has non-trivial potentials. (The situation is similar in quantum mechanics, where the ‘polymer particle representation’ in which the position operator has nice properties is better suited to deal with systems with general potentials [30].) If one chooses the second representation, the definition of the sub-space $\mathcal{H}_{V,\phi_o,\epsilon'}^{\text{scalar}}$ in the construction of the micro-canonical ensemble has to be modified suitably. These modifications are technically difficult but will not affect the final result.

3. It has been recently suggested [31] that one should use SO(3) rather than SU(2) as the group of internal rotations. In this case, to recover the Bekenstein-Hawking formula (1.1), the Barbero Immirzi parameter γ has to be set equal to $\gamma'_o = \ln 3/(2\pi\sqrt{2})$ in the minimally coupled case. Our calculation would then show that, in the non-minimally coupled case, the correctly modified expression (1.4) results for the *same* value γ'_o of γ . However, since the motivation behind the suggestion is somewhat ad-hoc and fails to be robust [32], this remark is meant only to be a mathematical observation.

4. In this paper, we restricted ourselves to type I horizons because we wanted to focus only on the modifications introduced by non-minimal coupling. Type II horizons can be included by combining this analysis with that of [23]. The main modification is that for type II horizons, one has to define the micro-canonical ensemble by specifying not just a constant ϕ_o but all the (invariantly defined) multipoles of ϕ at the horizon. This extension and technical issues contained in the foregoing remarks will be discussed elsewhere.

IV. DISCUSSION

Let us summarize. We found that in presence of non-minimal couplings, quantum geometry in the bulk undergoes a qualitative change because the triads, which dictate the Riemannian geometry in the bulk, now depend not just on the gravitational momentum $\hat{\Pi}^i$ but also on the scalar field $\hat{\phi}$. Eigenvalues of $\hat{\Pi}^i$ are discrete, in fact the same as those of the gravitational momentum $\hat{\Sigma}^i$ in the minimally coupled case. But the spectrum of $\hat{\phi}$ is continuous. Consequently, the kinematic arena provided by the bulk quantum geometry is significantly different in the Einstein frame from that in the Jordan frame. However, simplifications arise at the horizon because, *on the entire phase space*, the scalar field takes a fixed, constant value ϕ_o there. In particular, the triad operator smeared on S is simply rescaled by $f(\phi_o)$. Similarly, the only essential difference for the surface Hilbert space is that the expression of the level of the Chern-Simons theory is rescaled (from (2.3) to (3.5)). Consequently, the delicate matching between the bulk and surface theories required by the quantum boundary condition continues to be met. Using the same value (1.2) of the Barbero Immirzi parameter as in the minimally coupled case, for large black holes the statistical mechanical entropy is now given by $S = (f(\phi_o) a_o / 4 \ell_{\text{Pl}}^2)$ in the Jordan frame under consideration. In the case when $f(\phi)$ is everywhere positive on M , we could also have worked in the Einstein frame and the isolated horizon conditions would have been met again. Then, the analysis of [8] would have led us to the expression $\bar{S} = \bar{a}_o / 4 \ell_{\text{Pl}}^2$, where \bar{a}_o is the horizon area in the Einstein frame. However, from the relation between the two frames it follows that $\bar{a}_o = f(\phi_o) a_o$, whence the numerical value of entropy would be the same.

The calculations involved in our analysis are rather straightforward. Basically, there is now a new constant $f(\phi_o)$ and we have to keep track of how it modifies the analysis of [8]. However, the underlying conceptual issues are interesting. First, *a priori*, it was not clear that the Chern-Simons form of the surface symplectic structure would be preserved and the only modification would be in the expression of the level k . Secondly, because the action is now significantly different, it is far from being obvious that the geometrical, horizon boundary condition can be expressed in terms of the *canonical variables* a controllable fashion. It is a pleasant surprise that it *can be* so expressed and, furthermore, the modifications precisely compensate each other so that the quantum horizon condition continues to have solutions. Quantum geometry, developed in the mid-nineties [9-22], was based on the Hamiltonian framework where triads are canonically conjugate to the gravitational connection and Riemannian geometry can be built from *only* the gravitational sector of the phase space. From this perspective, the appearance of the scalar field in the expression of triads is a qualitative change. It could well have happened that the rather delicate matching required for a coherent theory of the quantum horizon falls apart in Jordan frames. The fact that the non-minimal coupling can be naturally accommodated, and the analysis leads to the entropy expression suggested classically by the first law, shows that the framework is robust.

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- [1] Ashtekar A, Beetle C, Dreyer O, Fairhurst S, Krishnan B, Lewandowski J and Wiśniewski J 2000 Generic Isolated Horizons and their Applications, *Phys. Rev. Lett.* **85** 3564
- [2] Ashtekar A, Fairhurst S and Krishnan B 2000 Isolated Horizons: Hamiltonian Evolution and the First Law *Phys. Rev.* **D62** 104025
- [3] Ashtekar A, Beetle C and Lewandowski J 2001 Mechanics of Rotating Isolated Horizons *Phys. Rev.* **D64** 044016
- [4] Ashtekar A, Beetle C and Lewandowski J 2002 Geometry of Generic Isolated Horizons, *Class. Quant. Grav.* **19** 1195-1225
- [5] Chrusciel P 1992 On the global structure of Robinson-Trautmann space-times *Proc. R. Soc.* **436** 299-316
- [6] Ashtekar A, Baez J, Corichi A and Krasnov K 1998 Quantum geometry and black hole entropy *Phys. Rev. Lett.* **80** 904-907
- [7] Ashtekar A, Corichi A and Krasnov K 1999 Isolated horizons: the classical phase space *Adv. Theor. Math. Phys.* **3** 419-471
- [8] Ashtekar A, Baez J, Krasnov K 2000 Quantum geometry of isolated horizons and black hole entropy *Adv. Theor. Math. Phys.* **4** 1-95
- [9] Ashtekar A and Isham C J 1992 Representation of the holonomy algebras of gravity and non-Abelian gauge theories *Class. Quant. Grav.* **9** 1433-1467
- [10] Ashtekar A and Lewandowski J 1994 Representation theory of analytic holonomy algebras *Knots and Quantum Gravity* ed Baez J C (Oxford: Oxford U. Press)
- [11] Baez J C 1994 Generalized measures in gauge theory *Lett. Math. Phys.* **31** 213-223
- [12] Ashtekar A and Lewandowski J 1995 Projective techniques and functional integration *Jour. Math. Phys.* **36** 2170-2191
- [13] Marolf D and Mourão J 1995 On the support of the Ashtekar-Lewandowski measure *Commun. Math. Phys.* **170** 583-606
- [14] Ashtekar A and Lewandowski J 1995 Differential geometry on the space of connections using projective techniques *Jour. Geo. & Phys.* **17** 191-230
- [15] Rovelli C and Smolin L 1995 Spin networks and quantum gravity *Phys. Rev.* **D52** 5743-5759
- [16] Baez J C 1996 Spin networks in non-perturbative quantum gravity *The Interface of Knots and Physics* ed Kauffman L (Providence, RI: American Mathematical Society, Providence) pp 167-203.
- [17] Ashtekar A, Lewandowski J, Marolf D, Mourão J and Thiemann T 1995 Quantization of diffeomorphism invariant theories of connections with local degrees of freedom *Jour. Math. Phys.* **36**, 6456-6493
- [18] Rovelli C and Smolin L 1995 Discreteness of area and volume in quantum gravity *Nucl. Phys.* **B442** 593-622; Erratum: *Nucl. Phys.* **B456**, 753
- [19] Ashtekar A and Lewandowski J 1997 Quantum theory of geometry I: Area operators *Class. Quant. Grav.* **14** A55-A81
- [20] Ashtekar A and Lewandowski J 1997 Quantum theory of geometry II: Volume Operators *Adv. Theor. Math. Phys.* **1** 388-429
- [21] Thiemann T 1998 A length operator for canonical quantum gravity *Jour. Math. Phys.* **39**

3372-3392

- [22] Rovelli C and Thiemann T 1998 The Immirzi parameter in quantum general relativity *Phys. Rev.* **D57** 1009
- Gambini R, Obregon O and Pullin J 1999 Yang-Mills analogues of the Immirzi ambiguity *Phys. Rev.* **D59** 047505
- [23] Ashtekar A 2003 http://www.phys.psu.edu/events/index.html?event_id=517
- [24] Jacobson T, Kang G and Myers R C 1994 On black hole entropy *Phys. Rev.* **D49** 6587-6598
- [25] Wald R M 1993 Black hole entropy is Noether charge *Phys. Rev.* **D48**, R3427-R3431
Iyer V and Wald R M 1994 Some properties of Noether charge and a proposal for dynamical black hole entropy *Phys. Rev.* **D50** 846-864
- [26] Ashtekar A, Corichi A and Sudarsky D 2003 Non-minimally coupled scalar fields and isolated horizons *Class. Quantum Grav.* **20** 3413-3426
- [27] Ashtekar A 1986 New variables for classical and quantum gravity *Phys. Rev. Lett.* **57** 2244–2247
Ashtekar A 1987 New Hamiltonian formulation of general relativity, *Phys. Rev.* **D36** 1587–1602.
- [28] Thiemann T 1998 Kinematical Hilbert Spaces for Fermionic and Higgs Quantum Field Theories *Class. Quant. Grav.* **15** 1487-1512
- [29] Ashtekar A, Lewandowski J and Sahlmann H 2003 Polymer and Fock representations of a scalar field *Class. Quantum Grav.* **20** L11-L21
- [30] Ashtekar A, Fairhurst S and Willis J 2003 Quantum gravity, shadow states, and quantum mechanics, *Class. Quantum Grav.* **20** 1031-1062
- [31] Dreyer O 2003 Quasinormal modes, the area spectrum, and black hole entropy *Phys. Rev. Lett.*, **90** 081301
- [32] Perez A, Sahlmann H and Sudarsky D 2003 communication to the ‘Spin Foam List’, April 16th.